CS46 practice problems 8

These practice problems are an opportunity for discussion and trying many different solutions. They are **not counted towards your grade**, and **you do not have to submit your solutions.** The purpose of these problems is to get more comfortable with reasoning and writing about Turing machines, decidability, and recognizability.

1. **Infinite languages.** Last week we saw that the following language was decidable:

$$\mathit{INFINITE}_{\mathrm{DFA}} = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) \text{ is an infinite language}\}$$

(See solved problem 4.10 in the book for a clever way of making this argument.)

(a) Show that $\mathit{INFINITE}_{\mathrm{CFG}}$ is decidable[1], where:

$$\mathit{INFINITE}_{\mathrm{CFG}} = \{\langle G \rangle \mid G \text{ is a context-free grammar and } L(G) \text{ is an infinite language}\}$$

(b) Show that $\mathit{INFINITE}_{\mathrm{TM}}$ is not decidable, where:

$$\mathit{INFINITE}_{\mathrm{TM}} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) \text{ is an infinite language}\}$$

2. **Equal language checking for grammars.**

$$\mathrm{EQ}_{\mathrm{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are context-free grammars and } L(G) = L(H)\}$$

(a) Show that $\mathrm{EQ}_{\mathrm{CFG}}$ is co-Turing-recognizable.[2]

(b) Show that $\mathrm{EQ}_{\mathrm{CFG}}$ is undecidable.[3]
(Note: this is why Automata Tutor for grammars ran thousands of test strings, instead of giving a definite answer!)

[1] Hint 1: The clever solution for $\mathit{INFINITE}_{\mathrm{DFA}}$ was linked to the pumping lemma for DFAs. How can you use the pumping lemma for context-free languages in a similar way?
Hint 2: The intersection of a context-free language with a regular language is context-free.

[2] Hint: Use nondeterminism.

[3] Hint: Theorem 5.13 shows $\mathrm{ALL}_{\mathrm{CFG}}$ is undecidable; you can use this result without proof.

Bonus problem if you finished the others:

Homomorphisms again! Recall the definition of homomorphism: A **homomorphism** is a function $f : \Sigma \to \Gamma^*$ from one alphabet to strings over another alphabet. We extend f to operate on strings by defining $f(w) = f(w_1)f(w_2)\cdots f(w_n)$ where $w = w_1w_2\cdots w_n$ and each $w_i \in \Sigma$. We further extend f to operate on languages by defining $f(\epsilon) = \epsilon$ and $f(A) = \{f(w) \mid w \in A\}$, for any language A.

(a) Show that the decidable languages are not closed under homomorphism. (That is, give an example language A and homomorphism f such that A is decidable, but $f(A)$ is not decidable.)

(b) A homomorphism is called **nonerasing** if it never maps a character to ε. (Equivalently, $|f(\sigma)| \geq 1$ for all $\sigma \in \Sigma$.) Prove that the decidable languages are closed under nonerasing homomorphisms.