Final thoughts on sets; DFAs

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Let's consider $\underline{\mathcal{P}(\mathbb{N})}$ ← the powerset of the natural numbers

(collection of all subsets of $\mathbb{N}$)

→ this is infinite: $\{1\}, \{2\}, \{3\}, \{4\}, \ldots \in \mathcal{P}(\mathbb{N})$

__bijection f:__

$1 \rightarrow \{1\}$

$2 \rightarrow \{2\}$

$3 \rightarrow \{3\}$

$\vdots$

$\{1, 2\}$

$\{1, 3\}$ . . . .

Seems like it has a problem: infinitely many sets of size 1, so we never get to the sets of size 2 . . .

__Fact__: $\mathcal{P}(\mathbb{N})$ is uncountable.

__Pf__: (by contradiction) Assume there exists a bijection $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

Let's use f to list all of $\mathcal{P}(\mathbb{N})$:

$f(1) = S_1$

$f(2) = S_2$

$f(3) = S_3$

$\vdots \quad \vdots$

We would like to find a contradiction by figuring out some set $D \subseteq \mathbb{N}$ which is not in this list.

So $D \in \mathcal{P}(\mathbb{N})$ but D is not mapped onto by f, so f is not onto, so f is not a bijection.

Let's define set D as follows: $D = \{ i \in \mathbb{N} \mid \underline{i \notin S_i} \}$.

Consider S_1.

If $1 \in S_1$, so $1 \notin D$. So $D \neq S_1$.

If $1 \notin S_1$, so $1 \in D$. So $D \neq S_1$.

Similarly, $S_2 \neq D$ (because of the number 2)

$S_3 \neq D$ (because of the number 3)

$\vdots$

So we have a set $D \subseteq \mathbb{N}$ but D is not in our list!

So f was not onto, so f was not a bijection. $\blacksquare$

This was another pf using DIAGONALIZATION (like when we showed $\mathbb{R}$ is uncountable):

	1	2	3	4	...
S_1	✓	×	✓	✓	
S_2	×	×	×	✓	...
S_3	✓	✓	×	×	...
S_4	×	×	✓	✓	...
$\vdots$					

DIAGONAL!

Our basic idea moving forward is to consider a computer as a machine that takes as input a STRING and as output says YES or NO.

Σ is a finite alphabet (\Sigma)

Σ^* is the set of all strings over Σ

<u>Def</u>: A <u>LANGUAGE</u> $L \subseteq \Sigma^*$ is a set of strings.

<u>Def</u>: Let M be a machine, then

$$L(M) = \{ w \in \Sigma^* \mid M \text{ accepts } w \}$$

says "YES"

We say $L(M)$ is the language <u>ACCEPTED</u> or <u>RECOGNIZED</u> by M.

In this class, Σ is finite.

What is $|\Sigma^*|$? definitely infinite, but countable (eg shortlex order)

Every computer program has a $\underbrace{\text{finite description}}_{\text{string}}$ over some Σ.

So there are countably infinitely many programs (each is some string in Σ^*).

How many languages are there? Each language is $\subseteq \Sigma^*$
so $\mathcal{P}(\Sigma^*)$ is the collection of all languages.

$$\underbrace{\left|\begin{array}{c}\text{the set of}\\\text{all programs}\end{array}\right|}_{\text{countable}} \neq \underbrace{\left|\begin{array}{c}\text{the set of}\\\text{all languages}\end{array}\right|}_{\text{uncountable}}$$

So there are some (infinitely many!) languages not recognized by ANY program!