Dovetailing and diagonalization

Wednesday, January 26, 2022 9:25

Def: Let S a set.

If there is a bijection $f: \mathbb{N} \to S$ then S is <u>countably infinite</u>.

If S is finite or countably infinite, then S is <u>countable</u>.

If S is not countable, it is <u>uncountable</u>.

Fact: The union of two countably infinite sets is countably infinite.

Pf: (directly)

Let $A = \{a_1, a_2, a_3, \dots\}$ be two countably infinite sets.

$B = \{b_1, b_2, b_3, \dots\}$

So we know there are two bijective functions $f_A: \mathbb{N} \to A$

$f_B: \mathbb{N} \to B$.

this is f_A: $1 \to a_1$, $2 \to a_2$, $3 \to a_3$, $\vdots$

f_B: $1 \to b_1$, $2 \to b_2$, $3 \to b_3$, $\vdots$

✗ This is not a set, it might have duplicates!

$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$

So let's build $f: \mathbb{N} \to A \cup B$: f: $1 \to a_1$, $2 \to b_1$, $3 \to a_2$, $4 \to b_2$, $\vdots$

Want to say f is a bijection.

<u>onto?</u> We can check and find any a_i or b_i in this list.

(because we're using f_A and f_B, which were both onto)

<u>one-to-one?</u> OOPS! If A and B have elements in common, these get mapped to twice.

Proposed fix: remove the duplicates.

(list them in f_B order, but renamed)

$B \setminus A =$ everything in B that's <u>not</u> in $A = \{c_1, c_2, c_3, \dots\}$

Now we can build f: $1 \to a_1$, $2 \to c_1$, $3 \to a_2$, $4 \to c_2$

<u>onto?</u> Yes, same reason as before. (we didn't remove too much)

<u>one-to-one?</u> Yes, we removed duplicates

$2 \to c_1$
$3 \to a_2$
$4 \to c_2$
$\vdots \quad \vdots$

one-to-one? Yes, we removed duplicates.

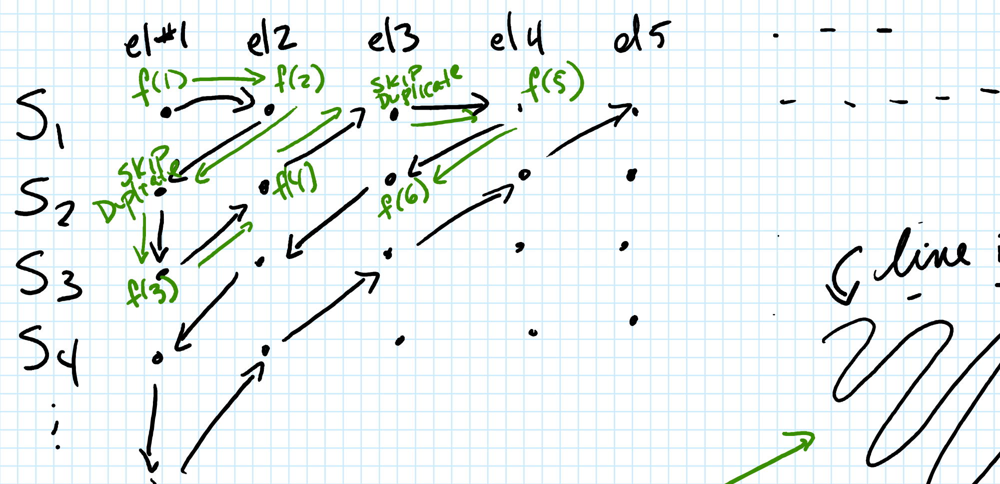
What about the union of 3 countably infinite sets? Still countably infinite.

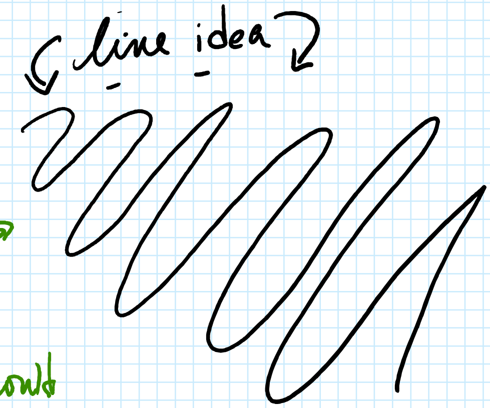
4? 5? 6? 7? :)

Fact: The union of countably infinitely many countably infinite sets is countably infinite.

Proof idea (by picture)

Let $S_1, S_2, S_3, \ldots$ be countably many countably infinite sets.





Build our bijection to follow this line but skip duplicates we've already covered, it should be onto & one-to-one.

This technique is called DOVETAILING.

The technique we saw Monday for "$|\mathbb{N}| \neq |\mathbb{R}|$" is called DIAGONALIZATION.

digits

x_1 500.$\boxed{0}$0000 . . .

$x_2 = \pi$ 3.1$\boxed{4}$159 . . .

x_1 $500.\boxed{0}0000$

$x_2 = \pi$ $3.1\boxed{4}159 \ldots$

$x_3 = e$ $2. \ldots \boxed{} \ldots$

$x_4 = 1/8$ $0.125\boxed{0}00 \ldots$

$\vdots$

follow the diagonal to build a number not in this list