Styles of proof:

- construction
  - e.g. Does a thing X exist?
    Answer: Yes, here is how to build it.

Thm: There exists an even number.

Pf (by construction):
2 is an even number.

- contradiction
- induction
- direct: just directly explain why the claim is true

Thm: If m and n are both perfect squares, then $m \cdot n$ is a perfect square.

Pf (direct): Suppose m and n are perfect squares.
Then there are some numbers x and y where $m = x^2$
$n = y^2$.

So $m \cdot n = x^2 \cdot y^2$ by substituting
$= x \cdot x \cdot y \cdot y$ def of squaring
$= (x \cdot y) \cdot (x \cdot y)$ by math
$= (x \cdot y)^2$ def of squaring

This means $m \cdot n$ is also a perfect square. ∎

Def: The CARDINALITY $|S|$ of a set S is the number of elements it contains.

e.g. $A = \{$cats, dogs$\}$ $B = \{$blue, purple$\}$

$|A| = 2 = |B|$

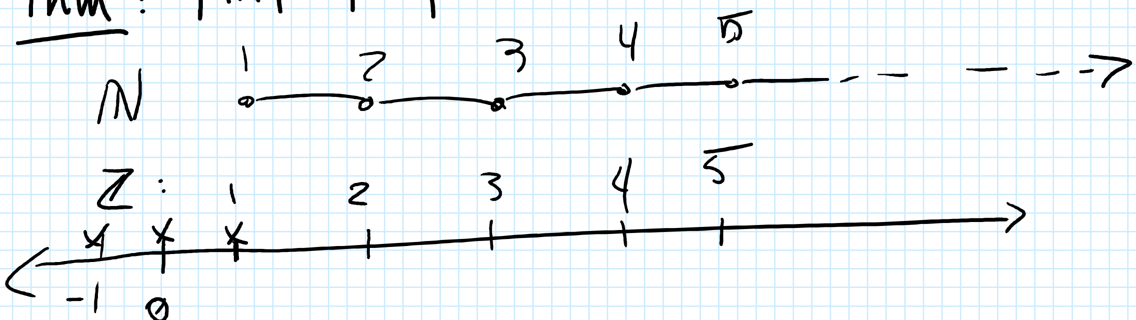
$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ $\quad |\mathbb{N}| =$ infinite

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ $\quad |\mathbb{Z}| =$ infinite

<u>Def</u>: Two sets A and B have the same cardinality $|A| = |B|$ if there is a function $f : A \to B$ which is $\underbrace{\text{one-to-one and onto}}_{\text{bijection}}$.

$$f: \begin{array}{ccc} \underline{A} & & \underline{B} \\ \text{cats} & \longrightarrow & \text{blue} \\ \text{dogs} & \longrightarrow & \text{purple} \end{array}$$

<u>Thm</u>: $|\mathbb{N}| = |\mathbb{Z}|$



<u>Pf</u>: by construction: want to build a bijection $f : \mathbb{N} \to \mathbb{Z}$

$$f: \begin{array}{ccc} \underline{\mathbb{N}} & & \underline{\mathbb{Z}} \\ 1 & \longrightarrow & 0 \\ 2 & \longrightarrow & 1 \\ 3 & \longrightarrow & -1 \\ 4 & \longrightarrow & 2 \\ 5 & \longrightarrow & -2 \\ \vdots & & \vdots \end{array}$$

<u>need to check</u>:

f one-to-one? yes

f onto? yes, we hit every element of $\mathbb{Z}$

How about $\mathbb{R}$? $\mathbb{R} = \{\ldots, -500, \pi, e, 6.758, 1, 1000.2, \ldots\}$

real numbers

$|\mathbb{R}|$ = infinite

Thm: $|\mathbb{N}| \neq |\mathbb{R}|$

Pf: by contradiction: Assume that $|\mathbb{N}| = |\mathbb{R}|$ so there exists a bijection $f: \mathbb{N} \to \mathbb{R}$. Argue why this is impossible.

We can write out f:

$$f: \begin{array}{l} 1 \longrightarrow x_1 \\ 2 \longrightarrow x_2 \\ 3 \longrightarrow x_3 \\ \vdots \qquad \vdots \end{array}$$

By assumption, f is onto so every real number is in the list $x_1, x_2, x_3, \ldots$.

We will come up with a number $r \in \mathbb{R}$ which is not in this list.

Let's build $r = 0.\_\,\_\,\_\,\_ \ldots$

pick a digit so that $r \neq x_1$

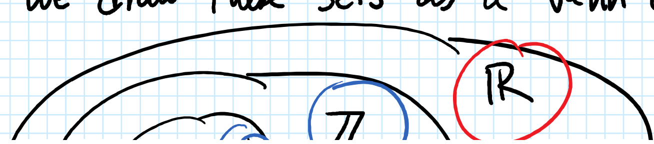
pick a digit so that $r \neq x_2$

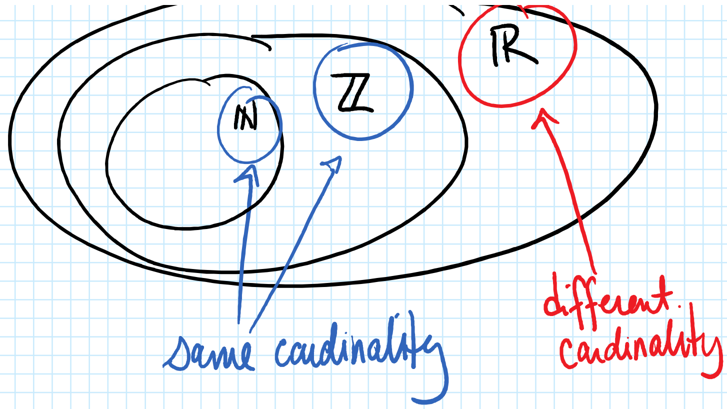
and so on: r's i^{th} digit after the decimal is different from x_i's i^{th} digit after the decimal, so $r \neq x_i$. 3

So f does not ever map onto r, $r \in \mathbb{R}$, so f is not onto! $\blacksquare$

Some final thoughts:

If we draw these sets as a Venn diagram, it looks like:





Our "usual" intuition about subsets & infinity doesn't always hold:

$\mathbb{N} \subsetneq \mathbb{Z}$, $\mathbb{N}$ is a subset of $\mathbb{Z}$ but same cardinality

$\mathbb{N} \subsetneq \mathbb{R}$, $\mathbb{N}$ is a subset of $\mathbb{R}$ but different cardinality

As notation (if you're interested), to distinguish these two sizes of infinity we write:

$|\mathbb{N}| = |\mathbb{Z}| = \aleph_0$ "aleph nought"

$|\mathbb{R}| = \aleph_1$ "aleph one"

... and we don't know if there's a cardinality between these!