4

DECIDABILITY

In Chapter 3 we introduced the Turing machine as a model of a general purpose computer and defined the notion of algorithm in terms of Turing machines by means of the Church–Turing thesis.

In this chapter we begin to investigate the power of algorithms to solve problems. We demonstrate certain problems that can be solved algorithmically and others that cannot. Our objective is to explore the limits of algorithmic solvability. You are probably familiar with solvability by algorithms because much of computer science is devoted to solving problems. The unsolvability of certain problems may come as a surprise.

Why should you study unsolvability? After all, showing that a problem is unsolvable doesn't appear to be of any use if you have to solve it. You need to study this phenomenon for two reasons. First, knowing when a problem is algorithmically unsolvable *is* useful because then you realize that the problem must be simplified or altered before you can find an algorithmic solution. Like any tool, computers have capabilities and limitations that must be appreciated if they are to be used well. The second reason is cultural. Even if you deal with problems that clearly are solvable, a glimpse of the unsolvable can stimulate your imagination and help you gain an important perspective on computation.

4.1 DECIDABLE LANGUAGES

In this section we give some examples of languages that are decidable by algorithms. We focus on languages concerning automata and grammars. For example, we present an algorithm that tests whether a string is a member of a context-free language (CFL). These languages are interesting for several reasons. First, certain problems of this kind are related to applications. This problem of testing whether a CFG generates a string is related to the problem of recognizing and compiling programs in a programming language. Second, certain other problems concerning automata and grammars are not decidable by algorithms. Starting with examples where decidability is possible helps you to appreciate the undecidable examples.

DECIDABLE PROBLEMS CONCERNING REGULAR LANGUAGES

We begin with certain computational problems concerning finite automata. We give algorithms for testing whether a finite automaton accepts a string, whether the language of a finite automaton is empty, and whether two finite automata are equivalent.

Note that we chose to represent various computational problems by languages. Doing so is convenient because we have already set up terminology for dealing with languages. For example, the ***acceptance problem*** for DFAs of testing whether a particular deterministic finite automaton accepts a given string can be expressed as a language, A_{DFA}. This language contains the encodings of all DFAs together with strings that the DFAs accept. Let

$$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}.$$

The problem of testing whether a DFA B accepts an input w is the same as the problem of testing whether $\langle B, w\rangle$ is a member of the language A_{DFA}. Similarly, we can formulate other computational problems in terms of testing membership in a language. Showing that the language is decidable is the same as showing that the computational problem is decidable.

In the following theorem we show that A_{DFA} is decidable. Hence this theorem shows that the problem of testing whether a given finite automaton accepts a given string is decidable.

THEOREM 4.1

A_{DFA} is a decidable language.

PROOF IDEA We simply need to present a TM M that decides A_{DFA}.

M = "On input $\langle B, w\rangle$, where B is a DFA and w is a string:
1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

PROOF We mention just a few implementation details of this proof. For those of you familiar with writing programs in any standard programming language, imagine how you would write a program to carry out the simulation.

First, let's examine the input $\langle B, w\rangle$. It is a representation of a DFA B together with a string w. One reasonable representation of B is simply a list of its five components: Q, Σ, δ, q_0, and F. When M receives its input, M first determines whether it properly represents a DFA B and a string w. If not, M rejects.

Then M carries out the simulation directly. It keeps track of B's current state and B's current position in the input w by writing this information down on its tape. Initially, B's current state is q_0 and B's current input position is the leftmost symbol of w. The states and position are updated according to the specified transition function δ. When M finishes processing the last symbol of w, M accepts the input if B is in an accepting state; M rejects the input if B is in a nonaccepting state.

We can prove a similar theorem for nondeterministic finite automata. Let

$$A_{\mathsf{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}.$$

THEOREM 4.2

A_{NFA} is a decidable language.

PROOF We present a TM N that decides A_{NFA}. We could design N to operate like M, simulating an NFA instead of a DFA. Instead, we'll do it differently to illustrate a new idea: Have N use M as a subroutine. Because M is designed to work with DFAs, N first converts the NFA it receives as input to a DFA before passing it to M.

N = "On input $\langle B, w\rangle$, where B is an NFA and w is a string:
1. Convert NFA B to an equivalent DFA C, using the procedure for this conversion given in Theorem 1.39.
2. Run TM M from Theorem 4.1 on input $\langle C, w\rangle$.
3. If M accepts, *accept*; otherwise, *reject*."

Running TM M in stage 2 means incorporating M into the design of N as a subprocedure.

Similarly, we can determine whether a regular expression generates a given string. Let $A_{\mathsf{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates string } w\}$.

THEOREM 4.3

A_{REX} is a decidable language.

PROOF The following TM P decides A_{REX}.

P = "On input $\langle R, w\rangle$, where R is a regular expression and w is a string:

1. Convert regular expression R to an equivalent NFA A by using the procedure for this conversion given in Theorem 1.54.
2. Run TM N on input $\langle A, w\rangle$.
3. If N accepts, *accept*; if N rejects, *reject*."

Theorems 4.1, 4.2, and 4.3 illustrate that, for decidability purposes, it is equivalent to present the Turing machine with a DFA, an NFA, or a regular expression because the machine can convert one form of encoding to another.

Now we turn to a different kind of problem concerning finite automata: *emptiness testing* for the language of a finite automaton. In the preceding three theorems we had to determine whether a finite automaton accepts a particular string. In the next proof we must determine whether or not a finite automaton accepts any strings at all. Let

$$E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}.$$

THEOREM 4.4

E_{DFA} is a decidable language.

PROOF A DFA accepts some string iff reaching an accept state from the start state by traveling along the arrows of the DFA is possible. To test this condition, we can design a TM T that uses a marking algorithm similar to that used in Example 3.23.

T = "On input $\langle A\rangle$, where A is a DFA:

1. Mark the start state of A.
2. Repeat until no new states get marked:
3. Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, *accept*; otherwise, *reject*."

The next theorem states that determining whether two DFAs recognize the same language is decidable. Let

$$EQ_{\mathsf{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}.$$

THEOREM 4.5

EQ_{DFA} is a decidable language.

PROOF To prove this theorem, we use Theorem 4.4. We construct a new DFA C from A and B, where C accepts only those strings that are accepted by either A or B but not by both. Thus, if A and B recognize the same language, C will accept nothing. The language of C is

$$L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right).$$

This expression is sometimes called the ***symmetric difference*** of $L(A)$ and $L(B)$ and is illustrated in the following figure. Here, $\overline{L(A)}$ is the complement of $L(A)$. The symmetric difference is useful here because $L(C) = \emptyset$ iff $L(A) = L(B)$. We can construct C from A and B with the constructions for proving the class of regular languages closed under complementation, union, and intersection. These constructions are algorithms that can be carried out by Turing machines. Once we have constructed C, we can use Theorem 4.4 to test whether $L(C)$ is empty. If it is empty, $L(A)$ and $L(B)$ must be equal.

F = "On input $\langle A, B\rangle$, where A and B are DFAs:

1. Construct DFA C as described.
2. Run TM T from Theorem 4.4 on input $\langle C\rangle$.
3. If T accepts, *accept*. If T rejects, *reject*."

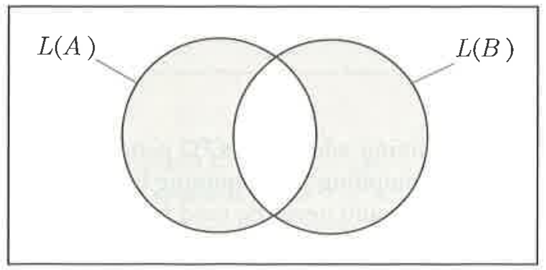


FIGURE 4.6
The symmetric difference of $L(A)$ and $L(B)$

DECIDABLE PROBLEMS CONCERNING CONTEXT-FREE LANGUAGES

Here, we describe algorithms to determine whether a CFG generates a particular string and to determine whether the language of a CFG is empty. Let

$$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}.$$

THEOREM 4.7

A_{CFG} is a decidable language.

PROOF IDEA For CFG G and string w, we want to determine whether G generates w. One idea is to use G to go through all derivations to determine whether any is a derivation of w. This idea doesn't work, as infinitely many derivations may have to be tried. If G does not generate w, this algorithm would never halt. This idea gives a Turing machine that is a recognizer, but not a decider, for A_{CFG}.

To make this Turing machine into a decider, we need to ensure that the algorithm tries only finitely many derivations. In Problem 2.26 (page 157) we showed that if G were in Chomsky normal form, any derivation of w has $2n - 1$ steps, where n is the length of w. In that case, checking only derivations with $2n - 1$ steps to determine whether G generates w would be sufficient. Only finitely many such derivations exist. We can convert G to Chomsky normal form by using the procedure given in Section 2.1.

PROOF The TM S for A_{CFG} follows.

S = "On input $\langle G, w\rangle$, where G is a CFG and w is a string:
1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n-1$ steps, where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

The problem of determining whether a CFG generates a particular string is related to the problem of compiling programming languages. The algorithm in TM S is very inefficient and would never be used in practice, but it is easy to describe and we aren't concerned with efficiency here. In Part Three of this book, we address issues concerning the running time and memory use of algorithms. In the proof of Theorem 7.16, we describe a more efficient algorithm for recognizing general context-free languages. Even greater efficiency is possible for recognizing deterministic context-free languages.

Recall that we have given procedures for converting back and forth between CFGs and PDAs in Theorem 2.20. Hence everything we say about the decidability of problems concerning CFGs applies equally well to PDAs.

Let's turn now to the emptiness testing problem for the language of a CFG. As we did for DFAs, we can show that the problem of determining whether a CFG generates any strings at all is decidable. Let

$$E_{\mathsf{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}.$$

THEOREM 4.8

E_{CFG} is a decidable language.

PROOF IDEA To find an algorithm for this problem, we might attempt to use TM S from Theorem 4.7. It states that we can test whether a CFG generates some particular string w. To determine whether $L(G) = \emptyset$, the algorithm might try going through all possible w's, one by one. But there are infinitely many w's to try, so this method could end up running forever. We need to take a different approach.

In order to determine whether the language of a grammar is empty, we need to test whether the start variable can generate a string of terminals. The algorithm does so by solving a more general problem. It determines *for each variable* whether that variable is capable of generating a string of terminals. When the algorithm has determined that a variable can generate some string of terminals, the algorithm keeps track of this information by placing a mark on that variable.

First, the algorithm marks all the terminal symbols in the grammar. Then, it scans all the rules of the grammar. If it ever finds a rule that permits some variable to be replaced by some string of symbols, all of which are already marked, the algorithm knows that this variable can be marked, too. The algorithm continues in this way until it cannot mark any additional variables. The TM R implements this algorithm.

PROOF

R = "On input $\langle G\rangle$, where G is a CFG:
1. Mark all terminal symbols in G.
2. Repeat until no new variables get marked:
3. Mark any variable A where G has a rule $A \rightarrow U_1U_2\cdots U_k$ and each symbol $U_1, \ldots, U_k$ has already been marked.
4. If the start variable is not marked, *accept*; otherwise, *reject*."

Next, we consider the problem of determining whether two context-free grammars generate the same language. Let

$$EQ_{\mathsf{CFG}} = \{\langle G, H\rangle |\ G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}.$$

Theorem 4.5 gave an algorithm that decides the analogous language EQ_{DFA} for finite automata. We used the decision procedure for E_{DFA} to prove that EQ_{DFA} is decidable. Because E_{CFG} also is decidable, you might think that we can use a similar strategy to prove that EQ_{CFG} is decidable. But something is wrong with this idea! The class of context-free languages is *not* closed under complementation or intersection, as you proved in Exercise 2.2. In fact, EQ_{CFG} is not decidable. The technique for proving so is presented in Chapter 5.

Now we show that context-free languages are decidable by Turing machines.

THEOREM 4.9

Every context-free language is decidable.

PROOF IDEA Let A be a CFL. Our objective is to show that A is decidable. One (bad) idea is to convert a PDA for A directly into a TM. That isn't hard to do because simulating a stack with the TM's more versatile tape is easy. The PDA for A may be nondeterministic, but that seems okay because we can convert it into a nondeterministic TM and we know that any nondeterministic TM can be converted into an equivalent deterministic TM. Yet, there is a difficulty. Some branches of the PDA's computation may go on forever, reading and writing the stack without ever halting. The simulating TM then would also have some non-halting branches in its computation, and so the TM would not be a decider. A different idea is necessary. Instead, we prove this theorem with the TM S that we designed in Theorem 4.7 to decide A_{CFG}.

PROOF Let G be a CFG for A and design a TM M_G that decides A. We build a copy of G into M_G. It works as follows.

M_G = "On input w:
1. Run TM S on input $\langle G, w\rangle$.
2. If this machine accepts, *accept*; if it rejects, *reject*."

Theorem 4.9 provides the final link in the relationship among the four main classes of languages that we have described so far: regular, context-free, decidable, and Turing-recognizable. Figure 4.10 depicts this relationship.

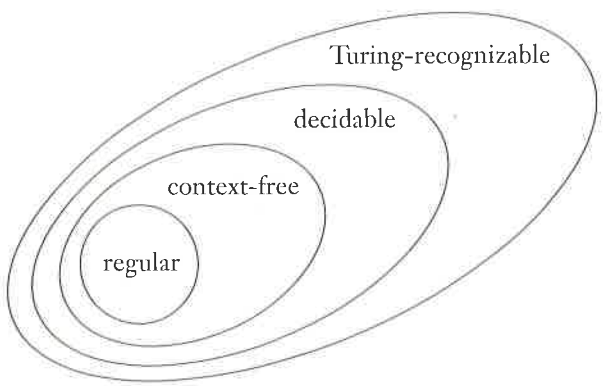


FIGURE 4.10
The relationship among classes of languages

4.2 UNDECIDABILITY

In this section, we prove one of the most philosophically important theorems of the theory of computation: There is a specific problem that is algorithmically unsolvable. Computers appear to be so powerful that you may believe that all problems will eventually yield to them. The theorem presented here demonstrates that computers are limited in a fundamental way.

What sorts of problems are unsolvable by computer? Are they esoteric, dwelling only in the minds of theoreticians? No! Even some ordinary problems that people want to solve turn out to be computationally unsolvable.

In one type of unsolvable problem, you are given a computer program and a precise specification of what that program is supposed to do (e.g., sort a list of numbers). You need to verify that the program performs as specified (i.e., that it is correct). Because both the program and the specification are mathematically precise objects, you hope to automate the process of verification by feeding these objects into a suitably programmed computer. However, you will be disappointed. The general problem of software verification is not solvable by computer.

In this section and in Chapter 5, you will encounter several computationally unsolvable problems. We aim to help you develop a feeling for the types of problems that are unsolvable and to learn techniques for proving unsolvability.

Now we turn to our first theorem that establishes the undecidability of a specific language: the problem of determining whether a Turing machine accepts a given input string. We call it A_{TM} by analogy with A_{DFA} and A_{CFG}. But, whereas A_{DFA} and A_{CFG} were decidable, A_{TM} is not. Let

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}.$$

THEOREM 4.11

A_{TM} is undecidable.

Before we get to the proof, let's first observe that A_{TM} is Turing-recognizable. Thus, this theorem shows that recognizers *are* more powerful than deciders. Requiring a TM to halt on all inputs restricts the kinds of languages that it can recognize. The following Turing machine U recognizes A_{TM}.

U = "On input $\langle M, w\rangle$, where M is a TM and w is a string:
1. Simulate M on input w.
2. If M ever enters its accept state, *accept*; if M ever enters its reject state, *reject*."

Note that this machine loops on input $\langle M, w\rangle$ if M loops on w, which is why this machine does not decide A_{TM}. If the algorithm had some way to determine that M was not halting on w, it could *reject* in this case. However, an algorithm has no way to make this determination, as we shall see.

The Turing machine U is interesting in its own right. It is an example of the *universal Turing machine* first proposed by Alan Turing in 1936. This machine is called universal because it is capable of simulating any other Turing machine from the description of that machine. The universal Turing machine played an important early role in the development of stored-program computers.

THE DIAGONALIZATION METHOD

The proof of the undecidability of A_{TM} uses a technique called *diagonalization*, discovered by mathematician Georg Cantor in 1873. Cantor was concerned with the problem of measuring the sizes of infinite sets. If we have two infinite sets, how can we tell whether one is larger than the other or whether they are of the same size? For finite sets, of course, answering these questions is easy. We simply count the elements in a finite set, and the resulting number is its size. But if we try to count the elements of an infinite set, we will never finish! So we can't use the counting method to determine the relative sizes of infinite sets.

For example, take the set of even integers and the set of all strings over $\{0,1\}$. Both sets are infinite and thus larger than any finite set, but is one of the two larger than the other? How can we compare their relative size?

Cantor proposed a rather nice solution to this problem. He observed that two finite sets have the same size if the elements of one set can be paired with the elements of the other set. This method compares the sizes without resorting to counting. We can extend this idea to infinite sets. Here it is more precisely.

DEFINITION 4.12

Assume that we have sets A and B and a function f from A to B. Say that f is ***one-to-one*** if it never maps two different elements to the same place—that is, if $f(a) \neq f(b)$ whenever $a \neq b$. Say that f is ***onto*** if it hits every element of B—that is, if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$. Say that A and B are the ***same size*** if there is a one-to-one, onto function $f\colon A \longrightarrow B$. A function that is both one-to-one and onto is called a ***correspondence***. In a correspondence, every element of A maps to a unique element of B and each element of B has a unique element of A mapping to it. A correspondence is simply a way of pairing the elements of A with the elements of B.

Alternative common terminology for these types of functions is ***injective*** for one-to-one, ***surjective*** for onto, and ***bijective*** for one-to-one and onto.

EXAMPLE 4.13

Let $\mathcal{N}$ be the set of natural numbers $\{1, 2, 3, \ldots\}$ and let $\mathcal{E}$ be the set of even natural numbers $\{2, 4, 6, \ldots\}$. Using Cantor's definition of size, we can see that $\mathcal{N}$ and $\mathcal{E}$ have the same size. The correspondence f mapping $\mathcal{N}$ to $\mathcal{E}$ is simply $f(n) = 2n$. We can visualize f more easily with the help of a table.

n	$f(n)$
1	2
2	4
3	6
$\vdots$	$\vdots$

Of course, this example seems bizarre. Intuitively, $\mathcal{E}$ seems smaller than $\mathcal{N}$ because $\mathcal{E}$ is a proper subset of $\mathcal{N}$. But pairing each member of $\mathcal{N}$ with its own member of $\mathcal{E}$ is possible, so we declare these two sets to be the same size. ∎

DEFINITION 4.14

A set A is ***countable*** if either it is finite or it has the same size as $\mathcal{N}$.

EXAMPLE 4.15

Now we turn to an even stranger example. If we let $\mathcal{Q} = \{\frac{m}{n} \mid m, n \in \mathcal{N}\}$ be the set of positive rational numbers, $\mathcal{Q}$ seems to be much larger than $\mathcal{N}$. Yet these two sets are the same size according to our definition. We give a correspondence with $\mathcal{N}$ to show that $\mathcal{Q}$ is countable. One easy way to do so is to list all the elements of $\mathcal{Q}$. Then we pair the first element on the list with the number 1 from $\mathcal{N}$, the second element on the list with the number 2 from $\mathcal{N}$, and so on. We must ensure that every member of $\mathcal{Q}$ appears only once on the list.

To get this list, we make an infinite matrix containing all the positive rational numbers, as shown in Figure 4.16. The ith row contains all numbers with numerator i and the jth column has all numbers with denominator j. So the number $\frac{i}{j}$ occurs in the ith row and jth column.

Now we turn this matrix into a list. One (bad) way to attempt it would be to begin the list with all the elements in the first row. That isn't a good approach because the first row is infinite, so the list would never get to the second row. Instead we list the elements on the diagonals, which are superimposed on the diagram, starting from the corner. The first diagonal contains the single element $\frac{1}{1}$, and the second diagonal contains the two elements $\frac{2}{1}$ and $\frac{1}{2}$. So the first three elements on the list are $\frac{1}{1}$, $\frac{2}{1}$, and $\frac{1}{2}$. In the third diagonal, a complication arises. It contains $\frac{3}{1}$, $\frac{2}{2}$, and $\frac{1}{3}$. If we simply added these to the list, we would repeat $\frac{1}{1} = \frac{2}{2}$. We avoid doing so by skipping an element when it would cause a repetition. So we add only the two new elements $\frac{3}{1}$ and $\frac{1}{3}$. Continuing in this way, we obtain a list of all the elements of $\mathcal{Q}$.

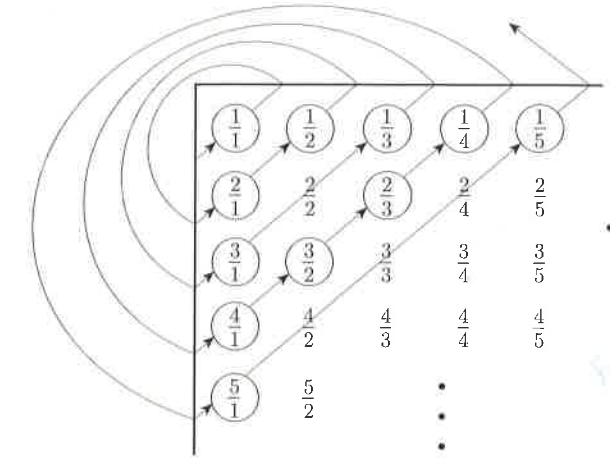


FIGURE 4.16
A correspondence of $\mathcal{N}$ and $\mathcal{Q}$

After seeing the correspondence of $\mathcal{N}$ and $\mathcal{Q}$, you might think that any two infinite sets can be shown to have the same size. After all, you need only demonstrate a correspondence, and this example shows that surprising correspondences do exist. However, for some infinite sets, no correspondence with $\mathcal{N}$ exists. These sets are simply too big. Such sets are called ***uncountable***.

The set of real numbers is an example of an uncountable set. A ***real number*** is one that has a decimal representation. The numbers $\pi = $ `3.1415926...` and

$\sqrt{2} = $ `1.4142135...` are examples of real numbers. Let $\mathcal{R}$ be the set of real numbers. Cantor proved that $\mathcal{R}$ is uncountable. In doing so, he introduced the diagonalization method.

THEOREM 4.17

$\mathcal{R}$ is uncountable.

PROOF In order to show that $\mathcal{R}$ is uncountable, we show that no correspondence exists between $\mathcal{N}$ and $\mathcal{R}$. The proof is by contradiction. Suppose that a correspondence f existed between $\mathcal{N}$ and $\mathcal{R}$. Our job is to show that f fails to work as it should. For it to be a correspondence, f must pair all the members of $\mathcal{N}$ with all the members of $\mathcal{R}$. But we will find an x in $\mathcal{R}$ that is not paired with anything in $\mathcal{N}$, which will be our contradiction.

The way we find this x is by actually constructing it. We choose each digit of x to make x different from one of the real numbers that is paired with an element of $\mathcal{N}$. In the end, we are sure that x is different from any real number that is paired.

We can illustrate this idea by giving an example. Suppose that the correspondence f exists. Let $f(1) = $ `3.14159...`, $f(2) = $ `55.55555...`, $f(3) = \ldots$, and so on, just to make up some values for f. Then f pairs the number 1 with `3.14159...`, the number 2 with `55.55555...`, and so on. The following table shows a few values of a hypothetical correspondence f between $\mathcal{N}$ and $\mathcal{R}$.

n	$f(n)$
1	3.14159...
2	55.55555...
3	0.12345...
4	0.50000...
⋮	⋮

We construct the desired x by giving its decimal representation. It is a number between 0 and 1, so all its significant digits are fractional digits following the decimal point. Our objective is to ensure that $x \neq f(n)$ for any n. To ensure that $x \neq f(1)$, we let the first digit of x be anything different from the first fractional digit 1 of $f(1) = $ `3.14159...`. Arbitrarily, we let it be 4. To ensure that $x \neq f(2)$, we let the second digit of x be anything different from the second fractional digit 5 of $f(2) = $ `55.555555...`. Arbitrarily, we let it be 6. The third fractional digit of $f(3) = $ `0.12345...` is 3, so we let x be anything different—say, 4. Continuing in this way down the diagonal of the table for f, we obtain all the digits of x, as shown in the following table. We know that x is not $f(n)$ for any n because it differs from $f(n)$ in the nth fractional digit. (A slight problem arises because certain numbers, such as `0.1999...` and `0.2000...`, are equal even though their decimal representations are different. We avoid this problem by never selecting the digits 0 or 9 when we construct x.)

n	$f(n)$
1	$3.\underline{1}4159\ldots$
2	$55.5\underline{5}555\ldots$
3	$0.12\underline{3}45\ldots$
4	$0.500\underline{0}0\ldots$
$\vdots$	$\vdots$

$x = 0.4641\ldots$

The preceding theorem has an important application to the theory of computation. It shows that some languages are not decidable or even Turing-recognizable, for the reason that there are uncountably many languages yet only countably many Turing machines. Because each Turing machine can recognize a single language and there are more languages than Turing machines, some languages are not recognized by any Turing machine. Such languages are not Turing-recognizable, as we state in the following corollary.

COROLLARY 4.18

Some languages are not Turing-recognizable.

PROOF To show that the set of all Turing machines is countable, we first observe that the set of all strings Σ^* is countable for any alphabet Σ. With only finitely many strings of each length, we may form a list of Σ^* by writing down all strings of length 0, length 1, length 2, and so on.

The set of all Turing machines is countable because each Turing machine M has an encoding into a string $\langle M \rangle$. If we simply omit those strings that are not legal encodings of Turing machines, we can obtain a list of all Turing machines.

To show that the set of all languages is uncountable, we first observe that the set of all infinite binary sequences is uncountable. An ***infinite binary sequence*** is an unending sequence of 0s and 1s. Let $\mathcal{B}$ be the set of all infinite binary sequences. We can show that $\mathcal{B}$ is uncountable by using a proof by diagonalization similar to the one we used in Theorem 4.17 to show that $\mathcal{R}$ is uncountable.

Let $\mathcal{L}$ be the set of all languages over alphabet Σ. We show that $\mathcal{L}$ is uncountable by giving a correspondence with $\mathcal{B}$, thus showing that the two sets are the same size. Let $\Sigma^* = \{s_1, s_2, s_3, \ldots\}$. Each language $A \in \mathcal{L}$ has a unique sequence in $\mathcal{B}$. The ith bit of that sequence is a 1 if $s_i \in A$ and is a 0 if $s_i \notin A$, which is called the ***characteristic sequence*** of A. For example, if A were the language of all strings starting with a 0 over the alphabet $\{0,1\}$, its characteristic sequence χ_A would be

$$
\begin{array}{rcccccccccccl}
\Sigma^* = \{ & \varepsilon, & 0, & 1, & 00, & 01, & 10, & 11, & 000, & 001, & \cdots & \} ; \\
A = \{ & & 0, & & 00, & 01, & & & 000, & 001, & \cdots & \} ; \\
\chi_A = & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & \cdots & & .
\end{array}
$$

The function $f\colon \mathcal{L} \longrightarrow \mathcal{B}$, where $f(A)$ equals the characteristic sequence of A, is one-to-one and onto, and hence is a correspondence. Therefore, as $\mathcal{B}$ is uncountable, $\mathcal{L}$ is uncountable as well.

Thus we have shown that the set of all languages cannot be put into a correspondence with the set of all Turing machines. We conclude that some languages are not recognized by any Turing machine.

AN UNDECIDABLE LANGUAGE

Now we are ready to prove Theorem 4.11, the undecidability of the language

$$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}.$$

PROOF We assume that A_{TM} is decidable and obtain a contradiction. Suppose that H is a decider for A_{TM}. On input $\langle M, w\rangle$, where M is a TM and w is a string, H halts and accepts if M accepts w. Furthermore, H halts and rejects if M fails to accept w. In other words, we assume that H is a TM, where

$$H(\langle M, w\rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ accepts } w \\ \textit{reject} & \text{if } M \text{ does not accept } w. \end{cases}$$

Now we construct a new Turing machine D with H as a subroutine. This new TM calls H to determine what M does when the input to M is its own description $\langle M \rangle$. Once D has determined this information, it does the opposite. That is, it rejects if M accepts and accepts if M does not accept. The following is a description of D.

D = "On input $\langle M \rangle$, where M is a TM:

1. Run H on input $\langle M, \langle M \rangle\rangle$.
2. Output the opposite of what H outputs. That is, if H accepts, *reject*; and if H rejects, *accept*."

Don't be confused by the notion of running a machine on its own description! That is similar to running a program with itself as input, something that does occasionally occur in practice. For example, a compiler is a program that translates other programs. A compiler for the language Python may itself be written in Python, so running that program on itself would make sense. In summary,

$$D(\langle M \rangle) = \begin{cases} \textit{accept} & \text{if } M \text{ does not accept } \langle M \rangle \\ \textit{reject} & \text{if } M \text{ accepts } \langle M \rangle. \end{cases}$$

What happens when we run D with its own description $\langle D \rangle$ as input? In that case, we get

$$D(\langle D \rangle) = \begin{cases} \textit{accept} & \text{if } D \text{ does not accept } \langle D \rangle \\ \textit{reject} & \text{if } D \text{ accepts } \langle D \rangle. \end{cases}$$

No matter what D does, it is forced to do the opposite, which is obviously a contradiction. Thus, neither TM D nor TM H can exist.

Let's review the steps of this proof. Assume that a TM H decides A_{TM}. Use H to build a TM D that takes an input $\langle M \rangle$, where D accepts its input $\langle M \rangle$ exactly when M does not accept its input $\langle M \rangle$. Finally, run D on itself. Thus, the machines take the following actions, with the last line being the contradiction.

- H accepts $\langle M, w \rangle$ exactly when M accepts w.
- D rejects $\langle M \rangle$ exactly when M accepts $\langle M \rangle$.
- D rejects $\langle D \rangle$ exactly when D accepts $\langle D \rangle$.

Where is the diagonalization in the proof of Theorem 4.11? It becomes apparent when you examine tables of behavior for TMs H and D. In these tables we list all TMs down the rows, $M_1, M_2, \ldots$, and all their descriptions across the columns, $\langle M_1 \rangle, \langle M_2 \rangle, \ldots$. The entries tell whether the machine in a given row accepts the input in a given column. The entry is *accept* if the machine accepts the input but is blank if it rejects or loops on that input. We made up the entries in the following figure to illustrate the idea.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*		*accept*		
M_2	*accept*	*accept*	*accept*	*accept*	
M_3					$\cdots$
M_4	*accept*	*accept*			
$\vdots$		$\vdots$			

FIGURE 4.19
Entry i, j is *accept* if M_i accepts $\langle M_j \rangle$

In the following figure, the entries are the results of running H on inputs corresponding to Figure 4.19. So if M_3 does not accept input $\langle M_2 \rangle$, the entry for row M_3 and column $\langle M_2 \rangle$ is *reject* because H rejects input $\langle M_3, \langle M_2 \rangle \rangle$.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*	
M_4	*accept*	*accept*	*reject*	*reject*	
$\vdots$		$\vdots$			

FIGURE 4.20
Entry i, j is the value of H on input $\langle M_i, \langle M_j \rangle \rangle$

In the following figure, we added D to Figure 4.20. By our assumption, H is a TM and so is D. Therefore, it must occur on the list $M_1, M_2, \ldots$ of all TMs. Note that D computes the opposite of the diagonal entries. The contradiction occurs at the point of the question mark where the entry must be the opposite of itself.

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$	$\cdots$
M_1	*accept*	*reject*	*accept*	*reject*		*accept*	
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*	$\cdots$
M_3	*reject*	*reject*	*reject*	*reject*		*reject*	
M_4	*accept*	*accept*	*reject*	*reject*		*accept*	
$\vdots$			$\vdots$		$\ddots$		
D	*reject*	*reject*	*accept*	*accept*		?	
$\vdots$			$\vdots$				$\ddots$

FIGURE 4.21
If D is in the figure, a contradiction occurs at "?"

A TURING-UNRECOGNIZABLE LANGUAGE

In the preceding section, we exhibited a language—namely, A_{TM}—that is undecidable. Now we exhibit a language that isn't even Turing-recognizable. Note that A_{TM} will not suffice for this purpose because we showed that A_{TM} is Turing-recognizable (page 202). The following theorem shows that if both a language and its complement are Turing-recognizable, the language is decidable. Hence for any undecidable language, either it or its complement is not Turing-recognizable. Recall that the complement of a language is the language consisting of all strings that are not in the language. We say that a language is ***co-Turing-recognizable*** if it is the complement of a Turing-recognizable language.

THEOREM 4.22

A language is decidable iff it is Turing-recognizable and co-Turing-recognizable.

In other words, a language is decidable exactly when both it and its complement are Turing-recognizable.

PROOF We have two directions to prove. First, if A is decidable, we can easily see that both A and its complement $\overline{A}$ are Turing-recognizable. Any decidable language is Turing-recognizable, and the complement of a decidable language also is decidable.

For the other direction, if both A and $\overline{A}$ are Turing-recognizable, we let M_1 be the recognizer for A and M_2 be the recognizer for $\overline{A}$. The following Turing machine M is a decider for A.

M = "On input w:
1. Run both M_1 and M_2 on input w in parallel.
2. If M_1 accepts, *accept*; if M_2 accepts, *reject*."

Running the two machines in parallel means that M has two tapes, one for simulating M_1 and the other for simulating M_2. In this case, M takes turns simulating one step of each machine, which continues until one of them accepts.

Now we show that M decides A. Every string w is either in A or $\overline{A}$. Therefore, either M_1 or M_2 must accept w. Because M halts whenever M_1 or M_2 accepts, M always halts and so it is a decider. Furthermore, it accepts all strings in A and rejects all strings not in A. So M is a decider for A, and thus A is decidable.

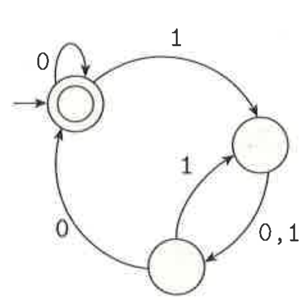
COROLLARY 4.23

$\overline{A_{\mathsf{TM}}}$ is not Turing-recognizable.

PROOF We know that A_{TM} is Turing-recognizable. If $\overline{A_{\mathsf{TM}}}$ also were Turing-recognizable, A_{TM} would be decidable. Theorem 4.11 tells us that A_{TM} is not decidable, so $\overline{A_{\mathsf{TM}}}$ must not be Turing-recognizable.

EXERCISES

[A]**4.1** Answer all parts for the following DFA M and give reasons for your answers.



a. Is $\langle M, 0100\rangle \in A_{\mathsf{DFA}}$?
b. Is $\langle M, 011\rangle \in A_{\mathsf{DFA}}$?
c. Is $\langle M\rangle \in A_{\mathsf{DFA}}$?
d. Is $\langle M, 0100\rangle \in A_{\mathsf{REX}}$?
e. Is $\langle M\rangle \in E_{\mathsf{DFA}}$?
f. Is $\langle M, M\rangle \in EQ_{\mathsf{DFA}}$?

4.2 Consider the problem of determining whether a DFA and a regular expression are equivalent. Express this problem as a language and show that it is decidable.

4.3 Let $ALL_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \Sigma^*\}$. Show that ALL_{DFA} is decidable.

4.4 Let $A\varepsilon_{\mathsf{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG that generates } \varepsilon\}$. Show that $A\varepsilon_{\mathsf{CFG}}$ is decidable.

[A]**4.5** Let $E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$. Show that $\overline{E_{\mathsf{TM}}}$, the complement of E_{TM}, is Turing-recognizable.

4.6 Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. We describe the functions $f\colon X \longrightarrow Y$ and $g\colon X \longrightarrow Y$ in the following tables. Answer each part and give a reason for each negative answer.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

n	$g(n)$
1	10
2	9
3	8
4	7
5	6

[A]**a.** Is f one-to-one?
b. Is f onto?
c. Is f a correspondence?
[A]**d.** Is g one-to-one?
e. Is g onto?
f. Is g a correspondence?

4.7 Let $\mathcal{B}$ be the set of all infinite sequences over $\{0,1\}$. Show that $\mathcal{B}$ is uncountable using a proof by diagonalization.

4.8 Let $T = \{(i, j, k) \mid i, j, k \in \mathcal{N}\}$. Show that T is countable.

4.9 Review the way that we define sets to be the same size in Definition 4.12 (page 203). Show that "is the same size" is an equivalence relation.

PROBLEMS

[A]**4.10** Let $INFINITE_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) \text{ is an infinite language}\}$. Show that $INFINITE_{\mathsf{DFA}}$ is decidable.

4.11 Let $INFINITE_{\mathsf{PDA}} = \{\langle M\rangle \mid M \text{ is a PDA and } L(M) \text{ is an infinite language}\}$. Show that $INFINITE_{\mathsf{PDA}}$ is decidable.

[A]**4.12** Let $A = \{\langle M\rangle \mid M \text{ is a DFA that doesn't accept any string containing an odd number of 1s}\}$. Show that A is decidable.

4.13 Let $A = \{\langle R, S\rangle \mid R \text{ and } S \text{ are regular expressions and } L(R) \subseteq L(S)\}$. Show that A is decidable.

[A]**4.14** Let $\Sigma = \{0,1\}$. Show that the problem of determining whether a CFG generates some string in 1^* is decidable. In other words, show that

$$\{\langle G\rangle \mid G \text{ is a CFG over } \{0,1\} \text{ and } 1^* \cap L(G) \neq \emptyset\}$$

is a decidable language.