CS41 Homework 4

This homework is due at 11:59PM on Sunday, October 1. Write your solution using LaTeX. Submit this homework using **github** as a **.tex** file. This is a **partnered homework**. You should primarily be discussing problems with your homework partner.

It's ok to discuss approaches at a high level with others. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else. The only exception to this rule is work you've done with other students *while in lab.* In this case, note (in your **homework submission poll**) who you've worked with and what parts were solved during lab.

1. **General asymptotic analysis.** For these problems, your example functions should have domain and range the positive integers $\mathbb{N}$.

   Let k be a fixed constant and suppose that $f_1, \ldots, f_k$ and h are functions such that $f_i = O(h)$ for all i.

   (a) Let $g_1(n) := f_1(n) + ... + f_k(n)$. Is $g_1 = O(h)$? Prove or give a counterexample. If you give a counterexample, give an alternate statement that best captures the relationship between the two functions.

   (b) Let $g_2(n) := f_1(n) \cdot \ldots \cdot f_k(n)$. Is $g_2 = O(h)$? Prove or give a counterexample. If you give a counterexample, give an alternate statement that best captures the relationship between the two functions.

2. **Who gets the loot?** A horde of seven barbarians has recently returned from pillaging the countryside, looting a hoard of 203 gold coins in the process. Now, they would like to divide their treasure. Being the enlightened barbarians they are, they have decided to vote on how to best divide the treasure.

   The barbarians' voting process is as follows. First, the strongest barbarian proposes a scheme for dividing the treasure. For example, she might propose to keep 143 gold coins for herself and give the six remaining barbarians 10 coins each. However, being the barbarians they are, if the majority vote against this scheme, then the rest of the barbarians kill the strongest. In this case, the strongest of the remaining six barbarians proposing a way to divide the treasure, risking death if more than half of the barbarians vote against him.

   The process repeats in a similar fashion (strongest remaining barbarian proposes a way to divide the hoard, barbarians vote, and the strongest remaining barbarian dies if they reject his suggestion) until a division is accepted.

   How should the strongest barbarian divide the hoard? You can assume that all barbarians are greedy and care only about how much treasure they receive.

   Note: If there are an even number of barbarians and a vote that is evenly split, the tie goes to the proposing barbarian (the other half of the barbarians are weaker and don't have the strength to kill the strongest). If a barbarian believes she will gain the same amount of gold whether or not the proposal is accepted, she will vote to reject.

3. **Paths in graphs.** A path P in a graph $G = (V, E)$ is a sequence of vertices $P = [v_1, \ldots, v_k]$ such that for all $1 \le i < k$ there is an edge $(v_i, v_{i+1}) \in E$. Path P is *simple* if all v_i's are distinct.

   In this problem, you will examine different graphs and consider how many different paths can exist in the graph.

   (a) Describe a graph G_1 on n vertices where between any two distinct vertices there are zero simple paths.

   (b) Describe a graph G_2 on n vertices where between any two distinct vertices there is exactly one simple path.

   (c) Describe a graph G_3 on n vertices where between any two distinct vertices there are exactly two simple paths.

   (d) **(extra challenge)** Describe a graph $G_4 = (V, E)$ on n vertices and two distinct vertices $s, t \in V$ such that there are $2^{\Omega(n)}$ simple $s \rightsquigarrow t$ paths.

4. **Rumor spreading.** The students who were taking college tours previously are now all at their colleges, and often chat and congregate online. After being admitted to their assigned (distinct) colleges, the same group of n students all go online to compare their experiences. One of them, s_1, wants to start a rumor that their college has ice cream at every meal, an on-campus rollercoaster, artisanal coffee, and no homework or exams, and is thus the *best* college, but s_1 wants to make sure that every other student will hear the rumor. Students always repeat rumors to their friends, but not all students are friends with all other students.

   If it takes one minute to repeat the rumor (copy-paste, plus time to pick an emoji and add a comment before forwarding), design and analyze an efficient algorithm which student s_1 can use to figure out whether every other student s_2, s_3, $\ldots$, s_n will hear the rumor. (Also, if they do all hear the rumor, then the algorithm should additionally report how long it will take until everyone has heard the rumor).

5. **(extra challenge)** For these problems, your example functions should have domain and range the positive integers $\mathbb{N}$.

   - Find (with proof) a function f_1 such that $f_1(2n)$ is $O(f_1(n))$.
   - Find (with proof) a function f_2 such that $f_2(2n)$ is not $O(f_2(n))$.

6. **(extra challenge)** For a positive integer k, call a graph k-colorable if the vertices can be properly colored using k colors. In other words, a bipartite graph is two-colorable. In this problem, you will investigate algorithms dealing with three-colorable graphs.

   (a) Design and analyze an algorithm which takes as input a graph $G = (V, E)$ and returns YES if G is three-colorable, and NO otherwise.

   (b) Design and analyze an efficient algorithm which takes as input a *three-colorable* graph $G = (V, E)$ and colors the vertices of the graph using $O(\sqrt{n})$ colors. (Note: while the input graph *is* three-colorable, it does not mean that we know what that coloring is!)