CS41 Lab 13: Approximation Algorithms

December 8 2022

This week, we'll continue exploring NP-COMPLETE decision problems, and develop approximation algorithms for related versions of those problems.

1. **Traveling Salesman Problem.** In this problem, a salesman travels the country making sales pitches. The salesman must visit n cities and then return to her home city, all while doing so as cheaply as possible.

   The input is a complete graph $G = (V, E)$ along with nonnegative edge costs $\{c_e : e \in E\}$. A *tour* is a simple cycle $(v_{j_1}, \ldots, v_{j_n}, v_{j_1})$ that visits every vertex exactly once.[1] The goal is to output the minimum-cost tour.

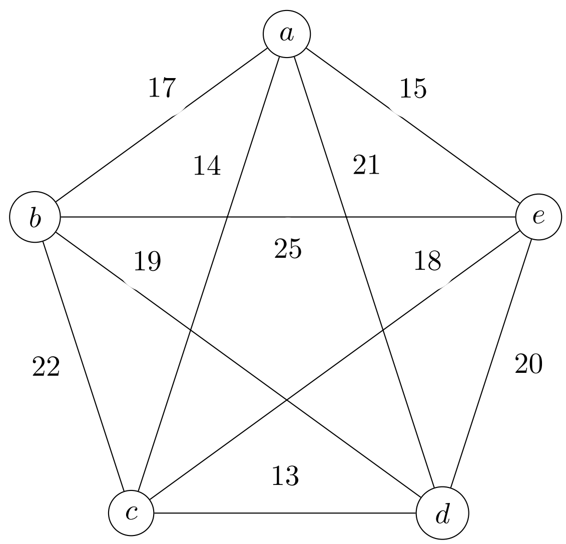
   For many TSP applications (such as when the cost is proportional to the distance between two cities), it makes sense for the edges to obey the *triangle inequality*: for every i, j, k, we have

   $$c_{(ik)} \leq c_{(ij)} + c_{(jk)}.$$

   This version is often called METRIC-TSP.

   The (decision version of the) Traveling Salesman Problem is NP-COMPLETE. For this problem, you will develop a 2-approximation algorithm for METRIC-TSP.

   (a) First, to gain some intuition, consider the following graph:

   (b) *On your own* try to identify a cheap tour of the graph.

   (c) Build some more intuition by computing the minimum spanning tree (MST) of the graph. Let T be your minimum spanning tree.

   (d) Let OPT be the cheapest tour. Show that its cost is bounded below by the cost of the MST: $\text{cost}(T) \leq \text{cost}(OPT)$.

[1] except for the start vertex, which we visit again to complete the cycle

(e) Give an algorithm which returns a tour A which costs at most twice the cost of the MST: $\text{cost}(A) \leq 2\,\text{cost}(T)$.

(f) Conclude that your algorithm is a 2-approximation for Metric-TSP.

2. **Toy-Storage**. William has lots of toys of all different sizes. You'd like to purchase a number of bins in which to store the toys. Approximately how many bins will you need?

   Let's formalize the Toy-Storage problem as follows. Suppose there are n toys, with sizes $s_1, \ldots, s_n$, with $0 < s_i < 1$ for all i. Assume each bin has size 1 and can hold any collection of toys whose total size is less than or equal to 1.

   In this problem, you'll develop a greedy approximation algorithm, which works by taking each toy in turn and placing it into the first bin that can hold it. Let $S := \sum_{i=1}^{n} s_i$.

   (a) Show that the optimal number of bins is at least $\lceil S \rceil$.

   (b) Show that the greedy algorithm leaves at most one bin half full.

   (c) Prove that the number of bins used by the greedy algorithm is at most $\lceil 2S \rceil$.

   (d) Prove that the greedy algorithm is a 2-approximation algorithm for the Toy-Storage problem.