CS41 Lab 12: NP Completeness

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This week, we've been discussing ways to classify problems according to their difficulty, using the notions of polynomial-time reductions and polynomial-time verifiers, and NP-Completeness. In this lab, you'll develop more sophisticated polynomial-time reductions using **gadgets**.

Below is a synopsis of relevant decision problems for this lab.

- SAT. The input for SAT is a set of n boolean variables $x_1, \ldots, x_n$ and m clauses $c_1, \ldots, c_m$, where each clause is the OR of one or more literals[1] e.g. $c_i = x_1 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_{17}$. Output YES iff there is a truth assignment to $x_1, \ldots, x_n$ that satisfies every clause.
- 3-SAT. The input for 3-SAT is the same as for SAT, except that each clause is the OR of exactly three literals.
- THREE-COLORING. The input for THREE-COLORING is a graph $G = (V, E)$. Output YES iff the vertices can be colored using three colors such that each edge has different-colored endpoints.

1. In the first exercise, you will reduce 3-SAT $\leq_P$ THREE-COLORING. Before getting there, it will be helpful to create some interesting three-colorable graphs. In all of the following exercises, you are to create a three-colorable graph (say the colors are red, blue, green) with certain special properties. The graphs you create should include three vertices marked a, b, c but can (and often will) include other vertices. Except for the properties specified, these vertices should be *unconstrained*. For example, unless the problem states that e.g. a cannot be red, it must be possible to color the graph in such a way that a is red. (You may fix colors for other vertices, just not a, b, c, and not in a way that constrains the colors of a, b, c.)
   (a) Create a graph such that a, b, c all have different colors.
   (b) Create a graph such that a, b, c all have the same color.
   (c) Create a graph such that a, b, c do NOT all have the same color.
   (d) Create a graph such that none of a, b, c can be green.
   (e) Create a graph such that none of a, b, c are green, and they cannot *all* be blue.
2. Show that 3-SAT $\leq_P$ THREE-COLORING. (Hint: Associate the color red with TRUE and the color blue with FALSE.)
3. Show that THREE-COLORING $\in$ NP-COMPLETE.
4. Show that SAT $\leq_P$ 3-SAT.

[1] A *literal* is either a boolean variable x_i or its negation $\bar{x}_i$.