CS41 Homework 9

This homework is due at 11:59PM on Sunday, November 24. Write your solution using LaTeX. Submit this homework in a file named `hw9.tex` using **github**. This is a **10 point assignment**. This is a partnered homework. You should primarily be discussing this homework with your partner.

It's ok to discuss approaches at a high level. In fact, I encourage you to discuss general strategies. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else. The only exception to this rule is work you've done with a lab partner/group *while in lab*. In this case, note (in your post-homework survey) who you've worked with and what parts were solved during lab.

The main **learning goals** of this homework are to deepen your understanding of reductions and polynomial-time reductions.

1. **Optimization vs Decision Problems**. Recall that a decision problem requires a YES/NO answer, and an optimization problem requires the "best possible answer", which often means maximizing or minimizing over some *cost* or *score*.

   For most optimization problems, there is an obvious analogue as a decision problem. For example, consider the following problem:

   VERTEX-COVER-OPT: Given a graph $G = (V, E)$, return the size of the smallest vertex cover in G.

   VERTEX-COVER-OPT has a natural decision problem, namely VERTEX-COVER. In fact, every optimization problem can be converted to a decision problem in this way.

   (a) Show that VERTEX-COVER $\leqslant_{\mathrm{P}}$ VERTEX-COVER-OPT.

   (b) Let B be an arbitrary optimization problem, and let A be the decision version of B. Show that

   $$A \leqslant_{\mathrm{P}} B \ .$$

   (c) Show that VERTEX-COVER-OPT $\leqslant_{\mathrm{P}}$ VERTEX-COVER.

2. **Liars and Friars.** To escape from some recent bad events, you sail off to a tropical island. This island is populated by n inhabitants; each inhabitant is either a *liar* or a *friar*. A friar always tells the truth, but liars cannot be trusted. You want to find the best place to eat dinner, and you definitely do not want to ask a liar (they will recommend bagel bar at Sharples, or Starbucks), so you would like to identify at least one friar. To help identify a friar, you can pair up any two inhabitants A, B and ask each to identify the other. If either A or B identifies that the other is a liar, then at least one of A and B is a liar. If both claim the other is a friar, then either both are liars or both are friars.

   (a) Show that if more than $n/2$ inhabitants are liars, it is not generally possible to identify a friar. You may assume liars collaborate to convince you they are friars.

   (b) Now, suppose that more than $n/2$ people are friars. Show that with at most $\lfloor \frac{n}{2} \rfloor$ pairwise comparisons, it is possible to reduce the problem to one of nearly half the original size. **Hint:** the algorithmic goal of this part is to produce an algorithm that, given n inhabitants (over half of whom tell the truth) returns a list of $\approx n/2$ inhabitants, over half of whom tell the truth.

(c) Use your solution to Part (2b) to construct an algorithm which identifies a single friar.

(d) Show how to find all friars, assuming that more than half of the n inhabitants are friars.

Note: Do not make any assumptions about n, e.g., do not assume n is odd, or n is a power of two, etc.

3. PATH-SELECTION (K&T 8.9) Consider the following problem. You are managing a communication network, modeled by a directed graph $G = (V, E)$. There are c users who are interested in making use of this network. User i (for each $i = 1, \ldots, c$) issues a *request* to reserve a specific path P_i in G on which to transmit data.

   You are interested in accepting as many of these path requests as possible, subject to the following restriction: if you accept both P_i and P_j, then P_i and P_j cannot share any nodes.

   Thus, the PATH-SELECTION problem asks: Given a directed graph $G = (V, E)$, set of requests $P_1, \ldots, P_c$—each of which is a path in G, and a number k, output YES iff it is possible to select at least k paths so that no two of the selected paths share any nodes.

   Show that INDEPENDENT-SET $\leqslant_P$ PATH-SELECTION.