# CS41 Homework 3

This homework is due at 11:59PM on Sunday, October 6. Write your solution using LaTeX. Submit this homework in a file named `hw3.tex` using **github**. This is a **two week, 18 point assignment**. This is a partnered homework. You should primarily be discussing this homework with your partner.

It's ok to discuss approaches at a high level. In fact, I encourage you to discuss general strategies. However, you should not reveal specific details of a solution, nor should you show your written solution to anyone else. The only exception to this rule is work you've done with a lab partner/group *while in lab*. In this case, note (in your post-homework survey) who you've worked with and what parts were solved during lab.

The main **learning goals** of this homework are to get more experience with asymptotic notation and to practice algorithm design skills for graph algorithms.

1. **Properties of Asymptotic Functions.** Let $k$ be a fixed constant and suppose that $f_1, \ldots,$ $f_k$ and $h$ are functions with domain and range $\mathbb{N}$ such that $f_i = O(h)$ for all $i$.

   (a) Let $g_1(n) := f_1(n) + ... + f_k(n)$. Is $g_1 = O(h)$? Prove or give a counterexample. If you give a counterexample, give an alternate statement that best captures the relationship between the two functions.

   (b) Let $g_2(n) := f_1(n) \cdot \ldots \cdot f_k(n)$. Is $g_2 = O(h)$? Prove or give a counterexample. If you give a counterexample, give an alternate statement that best captures the relationship between the two functions.

2. **Paths in Graphs.** A path $P$ in a graph $G = (V, E)$ is a sequence of vertices $P = [v_1, \ldots, v_k]$ such that for all $1 \leq i < k$ there is an edge $(v_i, v_{i+1}) \in E$. $P$ is *simple* if all $v_i$'s are distinct.

   In this problem, you will examine different graphs and consider how many different paths can exist in the graph.

   (a) Describe a graph $G_1$ on $n$ vertices where between any two distinct vertices there are zero simple paths.

   (b) Describe a graph $G_2$ on $n$ vertices where between any two distinct vertices there is exactly one simple path.

   (c) Describe a graph $G_3$ on $n$ vertices where between any two distinct vertices there are exactly two simple paths.

   (d) Describe a graph $G_4 = (V, E)$ on $n$ vertices and two distinct vertices $s, t \in V$ such that there are $2^{\Omega(n)}$ simple $s \rightsquigarrow t$ paths.

3. **(K&T 3.9)** There's a natural intuition that two nodes that are far apart in a communication network—separated by many hops—have a more tenuous connection than two nodes that are closer together. There are a number of algorithmic results that are based to some extent on different ways of making this notion precise. Here's one that involves the susceptibility of paths to the deletion of nodes.

   Suppose that an $n$-node undirected graph $G = (V, E)$ contains two nodes $s$ and $t$ such that the distance between $s$ and $t$ is strictly greater than $n/2$. (The **distance** between two nodes is the number of edges along the shortest path between them.)

(a) Show that there must exist some node $v$, not equal to either $s$ or $t$, such that deleting $v$ from $G$ destroys <u>*all*</u> paths $s \rightsquigarrow t$.

(b) Give an algorithm with running time $O(m+n)$ to find such a node $v$.

4. **Butterfly Classification(K& T 3.4)** Some of your friends are *lepidopterists* — they study butterflies. Part of their recent work involves collecting butterfly specimens and identifying what species they belong to. Unfortunately, determining distinct species can be difficult because many species look very similar to one another.

   During their last field expedition, your friends collected $n$ butterfly specimens and believe the specimens come from one of two butterfly species (call them species $A$ and $B$.) They'd like to divide the $n$ specimens into two groups—those that belong to $A$ and those that belong to $B$. However, it is very hard for them to directly label any one specimen. Instead, they adopt the following approach:

   For each pair of specimens $i$ and $j$, they study them carefully side by side. If they're confident enough in their judgement, they will label the pair as *same* (meaning they are confident that both specimens belong to the same species) of *different* (meaning they believe that the specimens belong to different species). If they are not confident, they do not label the specimens. Call this labeling (either $(i, j)$ are the same or $(i, j)$ are different) a *judgement.*

   A set of judgements is **consistent** if it is possible to label each specimen either $A$ or $B$ in such a away that for each pair $(i, j)$ labeled "same", it is the case that $i$ and $j$ have the same label, and for each pair $(i, j)$ labeled "different", it is the case that $i$ and $j$ have different labels.

   Design and analyze an algorithm which takes $n$ butterfly specimens and $m$ judgements, and outputs whether or not the judgements are consistent. Your algorithm should run in $O(n+m)$ time.

5. **Extra Credit.** Define a function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $f$ is $O(g)$ for all exponential functions $g$, but $f$ is *not* $O(h)$ for any polynomial function $h$.

6. **Extra Credit.** Is it possible to have two functions $f$ and $g$ such that $f$ is not $O(g)$ and $g$ is not $O(f)$? If so, give two example functions with this behavior; if not, prove that it is impossible for this to happen.