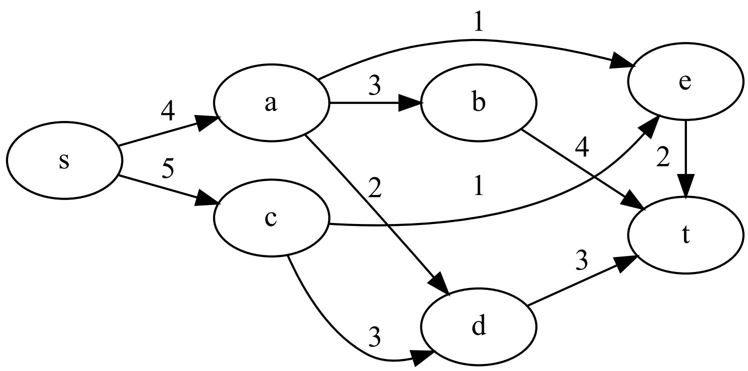
CS41 Lab 10: Network Flow

November 11, 2022

In typical labs this semester, you'll be working on a number of problems in groups of 3-4 students. You will not be handing in solutions; the primary purpose of these labs is to have a low-pressure space to discuss algorithm design. However, it will be common to have some overlap between lab exercises and homework sets.

The learning goals of this lab are to (i) get practice computing the maximum flow and (ii) practice using reductions to network flow.

1. **Maximum flow.** Find the maximum flow from s to t and the minimum cut between s and t in the network below. Show the residual graph at intermediate steps as you build the flow.



2. **Advertising contracts** (K& T 7.16)

   Back in the euphoric early days of the Web, people liked to claim that much of the enormous potential in a company like Yahoo! was in the "eyeballs"—the simple fact that millions of people look at its pages every day. Further, by convincing people to register personal data with the site, a site like Yahoo! can show each user an extremely targeted advertisement whenever the user visits the site, in a way that TV networks or magazines couldn't hope to match. So if a user has told Yahoo! that she is a 21-year-old computer science major at Swarthmore, the site can present a banner ad for apartments in Philadelphia suburbs; on the other hand, if she is a 50-year-old investment banker from Greenwich, CT, the site can display a banner ad pitching luxury cars instead.

   But deciding on which ads to show to which people involves some serious computation behind the scenes. Suppose that the managers of a popular site have identified k distinct *demographic groups* $G_1, G_2, \ldots, G_k$. (Some may overlap.) The site has contracts with m different advertisers to show a certain number of copies of their ads to users of the site. Here's what a contract with the i^{th} advertiser looks like:

   - For a subset $X_i \subseteq \{G_1, \ldots, G_k\}$ of the demographic groups, advertiser i wants ads shown only to users who belong to at least one of the groups listed in X_i.
   - For a number r_i, advertiser i want its ads shown to at least r_i users each minute.

   Consider the problem of designing a good *advertising policy* — a way to show a single ad to each user of the site. (Imagine a world where each user saw only *one* ad per site.) Suppose at a given minute, there are n users visiting the site. Because we have registration about each of the users, we know that user j belongs to a subset U_j of the demographic groups.

The problem is: is there a way to show a single ad to each user so that the site's contracts with each of the m advertisers is satisfied for this minute?

Give an efficient algorithm to decide if this is possible, and if so, to actually choose an ad to show each user.

3. **Bipartite Matching**

   Recall that a *bipartite graph* $G = (V, E)$ is an undirected graph where the vertices can be partitioned into $X, Y \subset V$, where $X \cap Y = \emptyset$ and $X \cup Y = V$, such that all edges have one end in X and the other in Y, that is $\forall (u, v) \in E$, $u \in X \wedge v \in Y$. A *matching* M in G is a subset of the edges $M \subseteq E$ such that each node appears in at most one edge of M. The *bipartite matching* problem is to find the matching M in bipartite graph G with maximum possible size. Give an efficient algorithm for bipartite matching.